Variable-wise and Term-wise Recentering

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Variable-wise recentering

- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$
- Let
  - $\Delta x_i = x_i - \mu_{x_i}$ for $i = 1,2$, with μ_{x_i} as arbitrary constants (perhaps a mean)

- Then the variable-wise recentered version is
  - $y = \beta_0^{\Delta} + \beta_1^{\Delta} \Delta x_1 + \beta_2^{\Delta} \Delta x_2 + \beta_{12}^{\Delta} \Delta x_1 \Delta x_2$

Term-wise recentering

- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$
- Let
  - $\Delta x_i = x_i - \mu_{x_i}$ for $i = 1,2$ and arbitrary μ_{x_i}.
  - $x_{12} = x_1 x_2$
  - $\Delta x_{12} = x_1 x_2 - \mu_{x_{12}}$, with $\mu_{x_{12}}$ as another arbitrary constant

- Then the term-wise recentered equation is
  - $y = \beta_0^{\text{tw}} + \beta_1^{\text{tw}} \Delta x_1 + \beta_2^{\text{tw}} \Delta x_2 + \beta_{12}^{\text{tw}} \Delta x_{12}$

Deriving variable-wise parameters

- $y = \beta_0^\Delta + \beta_1^\Delta \Delta x_1 + \beta_2^\Delta \Delta x_2 + \beta_{12}^\Delta \Delta x_1 \Delta x_2$

- Starting from the original equation and substituting
  - $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$
  - $y = \beta_0 + \beta_1(\Delta x_1 + \mu_{x_1}) + \beta_2(\Delta x_2 + \mu_{x_2}) + \beta_{12}(\Delta x_1 + \mu_{x_1})(\Delta x_2 + \mu_{x_2})$
  - $y = \left(\beta_0 + \beta_1 \mu_{x_1} + \beta_2 \mu_{x_2} + \beta_{12} \mu_{x_1} \mu_{x_2}\right) + \left(\beta_1 + \beta_{12} \mu_{x_2}\right)\Delta x_1 + \left(\beta_2 + \beta_{12} \mu_{x_1}\right)\Delta x_2 + \beta_{12} \Delta x_1 \Delta x_2$

- We get
  - $\beta_{12}^\Delta = \beta_{12}$
  - $\beta_1^\Delta = \beta_1 + \beta_{12} \mu_{x_2}$ and $\beta_2^\Delta = \beta_2 + \beta_{12} \mu_{x_1}$
  - $\beta_0^\Delta = \beta_0 + \beta_1 \mu_{x_1} + \beta_2 \mu_{x_2} + \beta_{12} \mu_{x_1} \mu_{x_2}$

Deriving term-wise parameters

- $y = \beta_0^{\mathrm{tw}} + \beta_1^{\mathrm{tw}}\Delta x_1 + \beta_2^{\mathrm{tw}}\Delta x_2 + \beta_{12}^{\mathrm{tw}}\Delta x_{12}$

- Starting from the original equation and substituting
  - $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$
  - $y = \beta_0 + \beta_1(\Delta x_1 + \mu_{x_1}) + \beta_2(\Delta x_2 + \mu_{x_2}) + \beta_{12}(\Delta x_{12} + \mu_{x_{12}})$
  - $y = \left(\beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} + \beta_{12}\mu_{x_{12}}\right) + \beta_1\Delta x_1 + \beta_2\Delta x_2 + \beta_{12}\Delta x_{12}$

- We get
  - $\beta_1^{tw} = \beta_1,\ \beta_2^{tw} = \beta_2,$ and $\beta_{12}^{tw} = \beta_{12}$
  - $\beta_0^{tw} = \beta_0 + \beta_1\mu_{x_1} + \beta_2\mu_{x_2} + \beta_{12}\mu_{x_{12}}$

- Which can also be written
  - $y = \beta_0^{tw} + \beta_1\Delta x_1 + \beta_2\Delta x_2 + \beta_{12}\Delta x_{12}$

Using the term-wise equation

- In the term-wise centered equation, the value of Δx_{12} depends on the values of Δx_1 and Δx_2.
    - $\Delta x_{12} = x_1 x_2 - \mu_{x_{12}} = (\Delta x_1 + \mu_{x_1})(\Delta x_2 + \mu_{x_2}) - \mu_{x_{12}}$
    - $\Delta x_{12} = \Delta x_1 \Delta x_2 + \Delta x_1 \mu_{x_2} + \Delta x_2 \mu_{x_1} + \mu_{x_1}\mu_{x_2} - \mu_{x_{12}}$

- Now the term-wise centered equation is equivalent to the variable-wise centered equation with the terms rearranged.
    - $y = \beta_0^{\text{tw}} + \beta_1^{\text{tw}} \Delta x_1 + \beta_2^{\text{tw}} \Delta x_2 + \beta_{12}^{\text{tw}} \Delta x_{12}$
    - $y = \beta_0^{tw} + \beta_1 \Delta x_1 + \beta_2 \Delta x_2 + \beta_{12} \Delta x_{12}$
    - $y = \beta_0^{tw} + \beta_1 \Delta x_1 + \beta_2 \Delta x_2 + \beta_{12}(\Delta x_1 \Delta x_2 + \Delta x_1 \mu_{x_2} + \Delta x_2 \mu_{x_1} + \mu_{x_1}\mu_{x_2} - \mu_{x_{12}})$

- We see that the final Δx_{12} term includes adjustments to Δx_1, Δx_2, and the constant. If we rearrange and simplify in the usual way, we arrive back at the variable-wise centered equation!
    - $y = (\beta_0^{tw} + \beta_{12}\mu_{x_1}\mu_{x_2} - \beta_{12}\mu_{x_{12}}) + (\beta_1 + \beta_{12}\mu_{x_2})\Delta x_1 + (\beta_2 + \beta_{12}\mu_{x_1})\Delta x_2 + \beta_{12}\Delta x_1 \Delta x_2$
    - [Now simplify the constant, and voila!]

As linear transformations

- $\beta^{\Delta} = C^{\Delta}\beta$
  - $\begin{bmatrix}\beta_0^{\Delta}\\ \beta_1^{\Delta}\\ \beta_2^{\Delta}\\ \beta_{12}^{\Delta}\end{bmatrix} = \begin{bmatrix}1 & \mu_1 & \mu_2 & \mu_1\mu_2\\ 0 & 1 & 0 & \mu_2\\ 0 & 0 & 1 & \mu_1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_{12}\end{bmatrix}$
- $\beta^{tw} = C^{tw}\beta$
  - $\begin{bmatrix}\beta_0^{\text{tw}}\\ \beta_1^{\text{tw}}\\ \beta_2^{\text{tw}}\\ \beta_{12}^{\text{tw}}\end{bmatrix} = \begin{bmatrix}1 & \mu_1 & \mu_2 & \mu_{12}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_{12}\end{bmatrix}$
- $\beta^{\Delta} = C^{t\Delta}\beta^{tw}$
  - $\begin{bmatrix}\beta_0^{\Delta}\\ \beta_1^{\Delta}\\ \beta_2^{\Delta}\\ \beta_{12}^{\Delta}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & \mu_1\mu_2 - \mu_{12}\\ 0 & 1 & 0 & \mu_2\\ 0 & 0 & 1 & \mu_1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\beta_0^{\text{tw}}\\ \beta_1^{\text{tw}}\\ \beta_2^{\text{tw}}\\ \beta_{12}^{\text{tw}}\end{bmatrix}$
  - Note $C^{t\Delta} = C^{\Delta}(C^{tw})^{-1}$